FROM THEORY TO METHODOLOGY: GUIDANCE FOR ANALYZING STUDENTS' COVARIATIONAL REASONING

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The fields of quantitative and covariational reasoning boast a wide range of powerful theoretical tools, which are described carefully in the literature. Less frequent and explicit attention, however, has been paid to writing down detailed, practical guidance for operationalizing these theoretical constructs. Some guidance is provided by covariational reasoning frameworks, but much is left unsaid concerning the inherent complexities and ambiguities involved in analyzing students' moment-by-moment behaviors and what these behaviors convey about their covariational reasoning. In an effort to more clearly link theory to analytic methodology, we share three lessons about analyzing students' covariational reasoning to make research more accessible to newcomers and better address what is often left unsaid in the covariational reasoning literature.

Keywords: Research Methods, Cognition, Precalculus, Calculus

The fields of quantitative and covariational reasoning boast a wide range of powerful theoretical tools. These tools have grown to be complex so they can more productively model students' thinking. The theoretical relationships between these elements are often detailed in theoretical synthesis papers, such as Thompson and Carlson (2017), and in the theoretical framework sections of empirical papers. But less frequent and explicit attention has been paid to writing down detailed, practical guidance for *operationalizing* these theoretical constructs for analysis. Experienced covariational reasoning researchers undoubtedly reflect on these important analytic considerations for each study they conduct; yet, these reflections are rarely reported in the literature.

In this paper, we build on prior work in which we reflected on and critiqued our analytic techniques for studying covariational reasoning to a) improve our own methodologies and analytic techniques and b) increase the accessibility of covariational reasoning research (Drimalla et al., 2020). In line with the theme of this year's PME-NA conference, we have decided to share our own productive struggles with designing and conducting a study that assessed students' covariational reasoning (Boyce et al., 2019). To center the *productive* aspect of these struggles, we will present the content of our reflections in the form of three lessons we have learned. Each of these lessons focus explicitly on developing analytic techniques and operationalizing theoretical constructs in quantitative and covariational reasoning research.

This paper is primarily intended for newcomers to covariational reasoning hoping to learn from our productive struggles. We also invite experts to engage with our experiences and critiques so that we can work together as covariational reasoning researchers to a) make research in this area more accessible to newcomers, b) be more open, clear, and explicit when writing about and sharing our analysis techniques and methodologies, and c) address potential flaws and

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gaps in the literature.

Theoretical Background

Covariational Reasoning

Carlson et al. (2002), building off Saldanha and Thompson's (1998) quantitative approach to covariational reasoning, described covariational reasoning to be "the cognitive activities involved in coordinating two varying quantities while attending to the way they change in relation to each other" (p. 354). Saldanha and Thompson (1998) further described how two quantities can be thought of simultaneously using the concept of a *multiplicative object*. They wrote, "Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two" (p. 299). Thus, covariational reasoning entails constructing two separate varying quantities as well as a multiplicative object, an object formed by simultaneously uniting the attributes of both quantities.

Quantitative reasoning. As covariational reasoning is a form of quantitative reasoning, the construction of quantities is foundational. Thompson (1994) described a *quantity* as a conceptual entity which "is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (pp. 7–8). For Thompson (2011) then, "quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit" (p. 37). Thus, quantitative reasoning is an individual's conception of quantities and their understanding of how the quantities relate. For example, a person could conceive of the height of an airplane, the distance it has traveled, and the relationship between the two.

Nonnormative graphing schemes. Covariational relationships are often represented graphically and, subsequently, the study of covariational reasoning is further complicated by the variety of ways students understand graphs. Graphing schemes that are the norm amongst mathematics education researchers can differ from students' graphing schemes (Moore et al., 2019). These nonnormative graphing schemes mainly stem from people's different meanings for coordinate systems (Lee et al., 2019), points in the coordinate system (Tasova & Moore, 2020), and curves with respect to the coordinate system (Moore & Thompson, 2015).

To attend to how students represent quantities graphically, Joshua et al. (2015) defined a *frame of reference* as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 32). In particular,

> an individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) *committing to a unit* so that all measures are multiplicative comparisons to it, 2) *committing to a reference point* that gives meaning to a zero measure and all non-zero measures, and 3) *committing to a directionality of measure comparison* additively, multiplicatively, or both. (p. 32, emphasis added)

Students who carry out each of these mental actions when representing a quantity, have constructed not just a quantity, but a framed quantity.

Operationalizing Theoretical Constructs for Covariational Reasoning

Our research group turned to two of the most widely-cited covariational reasoning frameworks—those of Carlson et al. (2002) and Thompson and Carlson (2017)—for guidance on operationalizing the above theoretical constructs to study students' covariational reasoning.

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Because this is where we suspect newcomers will begin, too, we start by attending to the combined guidance provided by these frameworks with regards to developing analytic frameworks for covariational reasoning. We then highlight the areas where the literature on these frameworks does not provide clear guidance.

Thompson and Carlson (2017) framework. Thompson and Carlson (2017) offered a framework of covariational reasoning levels based on the theoretical building blocks of quantity, multiplicative object, and variational reasoning. Each level is described based on what a person envisions while carrying out quantitative mental actions. Thompson and Carlson emphasized that when using this framework it is "essential to attend to how students are thinking that quantities' values vary and how they are uniting quantities' values when considering their meanings for covariation" (p. 443). To exemplify what it means to attend to these elements of student thinking, Thompson and Carlson shared examples of student approaches representative of each level of covariational reasoning for the classic Bottle Problem (Swan & Shell Centre, 1985) as well as a graphing task from Castillo-Garsow (2012).

The sample approaches to the Bottle Problem include one specific example of what a student at the gross coordination level might say; however, much of the discussion concerns what students at various levels might envision, focus on, or imagine—all internal processes. Less attention is paid to specific student behaviors that might be indicative of these internal processes.

For the graphing task, three kinds of student graphs are linked to specific covariational reasoning levels. Thompson and Carlson clarified that these graphical answers represent "at most" a certain level of reasoning because, for example, students "might have connected points simply because they thought that this is what one does when sketching a graph" (p. 442). In other words, they highlighted the need for researchers to carefully attend to the mental actions and images a student uses while creating a graph before attributing a corresponding level of covariational reasoning. Specific behaviors that might be indicative of these mental actions as students sketch their graph are not provided, however.

Carlson et al. (2002) framework. The levels of the Carlson et al. (2002) framework similarly highlight "images of covariation" that support particular mental actions. Unique to this framework is that each mental action is tied directly to 1–2 specific indicative behaviors. Some of the behaviors are involved in sketching a graph while others are what students say. Carlson et al. clarified that "Some students have been observed exhibiting behaviors that gave the appearance of engaging in [advanced mental actions] . . . When asked to provide a rationale for their construction, however, they indicated that they had relied on memorized facts to guide their construction" (pp. 361–362). As in Thompson and Carlson (2017)'s framework, then, each behavior corresponds to "at most" a particular level of covariational reasoning.

Critical Analysis of Both Frameworks. A primary benefit of the Thompson and Carlson (2017) framework is that it synthesizes several theoretical constructs found in the (co)variational reasoning literature. However, a newcomer using only this framework may struggle to operationalize these levels in their own research due to the lack of student behaviors tied explicitly to each level of covariational reasoning and the complexity of the theory. This reality reduces the analytic utility of this framework. On the other hand, the Carlson et al. (2002) framework provides specific graphical and verbal behaviors that correspond to each mental action/level but does not account for current theoretical advances in quantitative and covariational reasoning.

After reading the literature on both frameworks, newcomers may still wonder: What are behaviors I can observe that indicate a student has constructed and is (co)varying one or more

(framed) quantities? Also, as most examples of indicative behaviors from the literature on these frameworks focus on final products (e.g., the shape of a student's graph), how should one make sense of students' in-the-moment reasoning based on their constitutive behaviors that lead to these products? Such behaviors could be graphical or verbal—as in the Carlson et al. (2002) framework—but may also be gestural and inscriptional (either graphical or non-graphical).

Three Lessons for Analyzing Covariational Reasoning

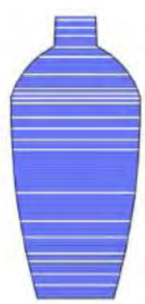
We asked similar questions while developing and refining our research group's methodologies for analyzing students' covariational reasoning. Our goal was to assess students' levels of covariational reasoning using the Thompson and Carlson (2017) framework, but this proved more difficult than we initially imagined. Although the literature details and carefully connects the theoretical aspects of quantitative and covariational reasoning, practical guidance for connecting this theory to analysis through task design and task analysis is much sparser.

Our Research Context

We learned the lessons we are about to share by reading the literature, through trial and error of analyzing undergraduate first semester calculus students' covariational reasoning for a prior study (Boyce et al., 2019), and through careful, systematic reflection both individually and as a research group (see Drimalla et al., 2020). Conversations with the last author—an expert in quantitative and covariational reasoning who was not a member of our research group—helped us frame and generalize our reflections to ensure that the lessons we learned apply beyond our experiences.

The covariational reasoning task we discuss in this paper is the Reverse Bottle Problem based on the classic Bottle Problem that has been used frequently in covariational reasoning research (Carlson et al., 2002; Paoletti & Moore, 2017; Stalvey & Vidakovic, 2015). The problem statement and the accompanying diagram we provided students is shown in Figure 1, alongside the main prompts we verbally asked participants.

To ground each lesson, we provide examples of student thinking. We focus on one student—Neal—in an attempt to accurately depict the challenges and ambiguities a researcher has to grapple with to analyze even one student's behaviors across a multi-part task.



Imagine this jug has been completely filled with water. It is then left indoors in a sunny window and left untouched until all the water has evaporated.

1) Describe how the height of the water in the jug would change as the volume of the water in the jug decreases.

2) Sketch a graph that gives the height of the liquid in the jug as a function of the volume of the water in the jug.

3) Describe the rate at which the height of the water is changing with respect to volume.

Figure 1: The Reverse Bottle Problem Task

Lesson 1: Use an Additional Analytic Frame to Track the Quantities Students Construct

Thompson's definition of quantity provides a helpful *theoretical* framing of what it means for students to construct a quantity. In practice, though, we found that an *analytic* frame for quantity is necessary to keep track of how students are conceptualizing and representing

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quantities on diagrams. Two primary difficulties necessitated the addition of an analytic frame for quantity.

First, we suspected that students used different words to refer to the same quantity they had constructed. However, our research team had no common method for gathering evidence that this was the case, leading to a hodge-podge of (implicit and potentially incompatible) perspectives being used, making consensus decisions difficult. A common question that arose is whether a student was simply reciting the words in the prompt or if they had truly constructed a variable quantity. We needed a way to make decisions on this front.

For example, Neal used the word “volume” six times when discussing the Bottle Problem in tandem with the initial diagram; however, he also used the phrase “amount of water” twice. Had Neal constructed two separate quantities? It was only after we used an analytic frame based on the language and gestures participants used that we noticed each gesture Neal used alongside “amount of water” was a subset of the gestures he used exclusively to refer to “volume.” This observation provided strong evidence that Neal was discussing the same quantity despite using different language. By saying “amount of water,” Neal used what we call *fresh* language—unique wording introduced by the interviewee, rather than the interview protocol or the interviewer—further cementing our conclusion that Neal was not merely repeating the word he expected the interviewer would want him to use in his response.

Second, the inclusion of volume necessitated that students construct a 3D quantity, but we initially had difficulty discerning how they represented such a quantity with fidelity when restricted to a 2D graph or diagram. Prior to the interview, Neal had encountered a problem in his class that involved a different shaped bottle being filled with water. The interviewer began by asking him to describe his recollection of that task. Neal initially described the bottle as “circular” (a 2D wording) but, within a few seconds, corrected himself and said that the bottle was “spherical” (a 3D wording). We credit this usage of fresh language as evidence of Neal’s awareness of dimensionality, which is key to understanding the quantities he constructs. Later, we again noted his awareness of dimensionality when Neal used unidimensional gestures to refer to most quantities aside from volume (e.g., “height” paired with a strictly vertical motion; “width” paired with a strictly horizontal motion). The primary gesture Neal used for volume (see Figure 2) appeared to be an attempt to represent the higher dimensionality of the volume quantity he had constructed in two or even three dimensions. This led us to conclude that he was not bound by the strictly 2D representation of the bottle when reasoning about his quantity for volume of the water in the jug.

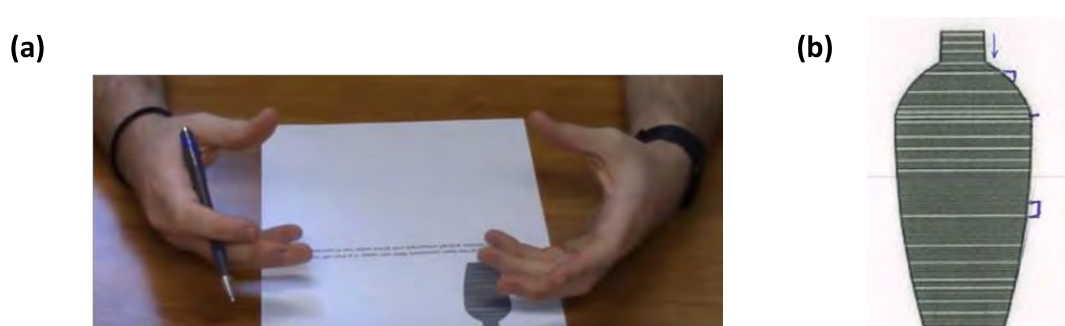

Figure 2: (a) Neal’s cupping gesture for volume (b) Neal’s bottle inscriptions

The lesson. The above two examples provide only a glimpse into the value our research

group found in adding an analytic frame to better understand how students conceptualized and communicated about the quantities they constructed. The lesson we learned is not that we needed to use this particular analytic framework for quantity but rather that choosing *an* analytic framework is essential for discerning a) whether a student has constructed a quantity, b) how a student conceptualizes a quantity, and c) how they communicate about their constructed quantity. Because such data are necessary to understand the extent to which students have varied or covaried quantities, we believe an analytic framework is essential for covariational reasoning research.

Lesson 2: Disentangle Graphing Schemes from Covariational Reasoning Schemes

Whenever graphical contexts are used to study someone's covariational reasoning, there is a risk of either over- or under-assessing that student's level of covariational reasoning. For example, some people can sketch a correct graph using their graphing schemes without relying on the highest levels of covariational reasoning (Carlson et al., 2002, pp. 361–362). Without careful analytic tools, a researcher may overassess such a person's capability for covariational reasoning. On the other extreme, students with highly nonnormative graphing schemes may struggle to graphically exhibit their highest level of covariational reasoning (Drimalla et al., 2020), resulting in an underassessment of their capability for covariational reasoning. This issue may persist even when people are asked to explain (their) graphs. Even Thompson (2016)—a seasoned covariational reasoning researcher—was surprised by the range of non-quantitative ways of thinking elicited by a graph explanation task that was carefully designed to assess covariational reasoning (pp. 448–450). These kinds of non-quantitative responses do not indicate a person *cannot* or *does not* reason covariationally; they merely indicate no present evidence of covariational reasoning.

We ran headfirst into these methodological issues when assessing Neal's covariational reasoning level based on his graphical activity. Previously, Neal had spent just under 2.5 minutes establishing a quantitative frame (Moore & Carlson, 2012) and discussing the covariational relationship between the height and volume of water using the provided diagram (See Figure 1). On the other hand, graphing this relationship and crafting an explanation of the graph that Neal found satisfactory and consistent with his prior diagram-based explanation took approximately 20 minutes. We suspect much of this time was spent grappling with nonnormative graphing schemes and frames of reference for the quantities of height and volume, rather than exhibiting Neal's ability to reason covariationally.

Directionality of the x-axis and conventions for slope. Neal first labeled his x-axis with "volume water in jug" and his y-axis with "height of water" then added the words "empty" and "full" to fix a directionality of measure along both axes consistent with normative conventions (i.e., on the x-axis, quantities increase from left to right; on the y-axis, quantities increase from bottom to top). Next, Neal plotted the initial point of his graph at (empty, full) because "the height of the water is at the maximum . . . and we want to start at the maximum point." Moments later, though, he realized that the initial point should be (full, full) and flipped the directionality of his volume axis using labels of "full" and "0" to make it so.

Neal then sketched a curve with the correct shape. However, his justifications for this shape were based almost entirely on "the rate" without mention of any quantities. The interviewer asked Neal to provide a value for the rate of change of height with respect to volume using his graph. Neal explained that the rate of change in the beginning is -1 because "every time it moves down one it goes over one." Here, Neal used the usual convention for calculating slope without attending to the nonnormative directionality of his x-axis. He justified his claim by citing that the

graph is decreasing, but then experienced a perturbation, claiming "that's weird," and noting that his x-axis decreased from left to right. But it was not until after the interviewer prompted Neal to add units on each axis that he performed a rise over run calculation and concluded that the slope was, indeed, positive, overriding the normative convention that a graph that "goes down" from left to right must have a negative slope.

The lesson. Neal was one of the only study participants who eventually overrode his normative graphing schemes to justify the graph he drew. Clearly, this is no easy feat, as it took Neal nearly 20 minutes to adapt to these nonnormative conditions and draw covariational conclusions he agreed were consistent with what he had deduced using the bottle diagram.

From this, we learned how important it is to have built-in methodology or an analytic tool to distinguish people's quantitative and covariational reasoning schemes from graphing schemes based on memorized conventions. There are a few ways this could be accomplished: 1) Establish a baseline for a participant's idiosyncratic graphing schemes earlier in the interview to contextualize their covariational reasoning schemes on tasks that likely invoke nonnormative graphing schemes. This could be done using careful task design that gradually increases in difficulty and the extent to which nonnormative graphing schemes likely need to be used. 2) Use an analytic framework for quantity in the graphical context to better foreground any quantitative reasoning that does occur relative to students' (possibly nonnormative) frames of reference and graphing schemes. This approach mirrors Lesson 1.

In our case, we went with option 2 and attended carefully to how Neal represented relevant quantities graphically (in normative and nonnormative ways) using the frames of reference analytic framework. This additional analytic framework enabled us to observe that only after Neal committed to a unit for both height and volume did he recognize that the slope of his graph was actually positive all along. Although Neal had constructed *partially* framed quantities for height and volume early on, it was not until he constructed fully framed quantities that he stopped relying on normative graphing schemes and began to reason with the quantities he had previously represented along the axes.

Lesson 3: Attend to Non-Graphical Behaviors Indicative of Covariational Reasoning

After becoming aware of the delicate issue of using graphs to assess covariational reasoning in Lesson 2 (see also Drimalla et al., 2020), we sought non-graphical behaviors that could be indicative of covariational reasoning. Primarily, we investigated participants' interactions with the bottle diagram (see Figure 1) when they were prompted to describe how the height of the water in the jug would change as the volume of the water in the jug decreases. Often, participants would instead discuss how only one of these quantities varies with time, rather than how the quantities themselves covary. To assess participants' capacity for covariational reasoning, though, we wanted to be as certain as possible that the quantities a student was envisioning as covarying were the water's height and volume. Thus, we distinguish between conceptual time, which is a constructed quantity, and experiential time, which is not. This distinction—of explicitly making a measured note of time—is relevant for understanding individuals' reasoning about multiple quantities (Thompson & Carlson, 2017) since conceptual time is sometimes included as one of the quantities to covary. After all, varying just one quantity (with respect to experiential time) is not **co**variational reasoning. And even if a participant had been covarying a quantity with *conceptual* time, we had no means to determine whether they had constructed time as a quantity. In the following section, we analyze some of Neal's tracing actions and wonder out loud whether his actions are sufficient evidence of covariational reasoning.

Tracing the side of the bottle. After Neal had established some typical gestures for both

height and volume, he traced downwards along the final stretch of the right side of the bottle while simultaneously describing the relationship between the height and volume in the final portion of the bottle: “When the jug starts to get thinner the height at which—er, the *rate* at which the height is decreasing is going to raise.” Next, Neal singled out a segment of the bottle in the section he had just traced along by drawing a bracket (see Figure 2b) and justified his prior answer: “Because the volume of water which is decreasing at a constant rate will take smaller—uh, sorry—*larger* and *larger* heights [traces vertically down bracket] to contain that equal amount of volume [cups hands in volume gesture, see Figure 2a].” The fluency and simultaneity of Neal’s actions, gestures, and verbal explanation led us to believe that these behaviors were evidence Neal had engaged in continuous covariational reasoning. In particular, Neal’s selection of an arbitrary sub-interval of bottle in which he discussed that the same volume would take up “larger and larger heights” led us to believe he had envisioned these quantities as having values which change. This seems to indicate Neal was engaging in more than gross covariational reasoning.

But was this sufficient evidence? Alternatively, perhaps Neal was merely using this tracing action as a way to track experiential time. When later drawing the shape of his graph, he often re-traced the side of the bottle, potentially envisioning what the water in the bottle looks like as time passes. Was Neal actually envisioning the values of height and volume as covarying, or just the gross variation in each quantity with respect to experiential time? As we remarked earlier, it was not until the interviewer suggested Neal add a scale to his axes (about 15 minutes into graphing) that he explicitly established units for each quantity. Had he implicitly been thinking in units as far back as when he was first tracing the bottle diagram? Possibly. But there is certainly some ambiguity present which affects the conclusions we can draw from his earlier tracing behavior.

The lesson. In this lesson, we sought to highlight the difficulty of drawing covariational conclusions from a person’s observable behaviors. We believe there is great value in attending to non-graphical behaviors to assess covariational reasoning—both because it helps make the theory of covariational reasoning more accessible to newcomers but also because it expands the methodologies available to study covariational reasoning. At present, much of the literature only reports briefly or implicitly on the many ambiguities present in interpreting these behaviors. We hope this changes and encourage other covariational reasoning researchers to make this a reality.

For newcomers hoping to study covariational reasoning *now*, we envision two approaches to help resolve such ambiguities in observable behaviors. First, plan to ask targeted follow-up questions in your interview protocol to clarify whether participants are reasoning about quantities’ values. In our case, we had already conducted interviews and could not do this, but in retrospect we suspect that asking Neal a simple follow-up question after he had traced the bottle may have resolved our uncertainty. Second, incorporate additional subtasks to ascertain whether participants are reasoning about quantities’ values. To support this endeavor, there is research concerning task design and sequencing in covariational reasoning research (e.g., Johnson, 2015).

Conclusion

By sharing these lessons, we hope to have drawn attention to the unwritten complexities of analyzing students’ moment-by-moment behaviors while studying their covariational reasoning. Namely, the importance of attending carefully to the quantities people construct (Lessons 1 & 2), the challenge of disentangling normative graphing schemes from covariational reasoning schemes (Lesson 2), and the ambiguities in interpreting non-graphical behaviors as indicative of covariational reasoning (Lesson 3). Given the long history and complex theoretical nature of

quantitative/covariational reasoning research, such conversations are *essential* if we wish to keep this subfield accessible, rather than daunting, to newcomers. We hope we have contributed to this end; however, we cannot do this alone. We welcome others—experts and *especially* newcomers—to write explicitly on these topics and forge more detailed connections between theory and methodology in the covariational reasoning literature.

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